First summer school within the framework of the

International Doctoral Program in Mathematics at TSU

Operator Algebras, Spectral Theory, and Topological Insulators

Tbilisi, September 17-21, 2018

Titles and Abstracts

Malkhaz Bakuradze (Tbilisi State University): Chern characters and K-theory.

Abstract We will provide a brief introduction to the formal theory of characteristic classes. We will consider carefully the case of Chern classes and de Rham cohomology with the aim to understand how a connection on a vector bundle E over a manifold M can be represented locally by a matrix of 1-forms. Similarly, the curvature of the connection can be represented by a matrix of 2-forms. Under a change of local frame, the curvature matrix Ω transforms by conjugation. Thus, if X is a matrix of size $r \times r$ and $P(X)$ is a polynomial in r^2 variables invariant under conjugation by the elements of the general linear group, then the differential form $P(\Omega)$ will be independent of the frame and will define a global form on M. This global form $P(\Omega)$ is closed and is independent of the connection. For a vector bundle E, the cohomology class $[P(\Omega)]$ is therefore a well-defined element of the de Rham cohomology of M depending only on the invariant polynomial $P(X)$. This gives rise to an algebra homomorphism called the Chern-Weil homomorphism from the algebra of invariant polynomials on Lie algebra of general linear group to the de Rham cohomology algebra of M. For each homogeneous invariant polynomial $P(X)$ of degree k, the cohomology class $[P(\Omega)] \in H^{2k}(M)$ is an isomorphism invariant of the vector bundle. In this sense, the class $[P(\Omega)]$ is called a characteristic class of E.

Further reading

[1] L. W. Tu, *Differential geometry: Connections, curvature, and characteristic classes*, Grad. Texts Math., vol. 275, Springer, Cham, 2017.

Gian Michele Graf (ETH Zurich): Anderson localization and topological phases.

Abstract The first lecture is meant to provide a bird's eye view on the topics covered. We will begin by recalling the physical origin of topological insulators, to be found in the (2-dimensional) Integer Quantum Hall effect (IQHE), and by emphasizing the role of disorder due to Anderson localization; a more thorough introduction to which will be provided in the lectures by C. Rojas-Molina. We will discuss different interpretations of the quantum Hall conductance and see how they are heuristically equivalent, for instance by the bulk-edge correspondence. A more modern view of topological insulators, based on symmetries in various dimensions, will also be given.

In the second lecture we will see, in the case of the IQHE, how the different interpretations prompt different mathematical indices and why they are equal.

In the third lecture we will temporarily drop disorder to see how such an index reduces to the Chern number of a certain vector bundle. The relevant index in presence of time-reversal symmetry (Fu-Kane index) will be introduced as well.

In the fourth lecture we will look at the disordered chain with chiral symmetry, define its indices, and discuss bulk-edge correspondence.

In the last lecture we will discuss Floquet insulators, where time comes in as a supplementary dimension.

Ralf Meyer (University of Göttingen): From materials to K-theory and operator algebras.

Abstract I will start with a crash course in quantum mechanics, motivating why electronic properties of a crystal may be modeled by matrix-valued functions on a torus. Some aspects of [2] will be mentioned to clarify the link to physics. A topological phase is an equivalence class of such matrix-valued functions, and the equivalence classes form a K-theory group. Extra symmetries such as a time-reversal symmetry modify which K-theory group is relevant. An important point here is to choose the space or the C^*-algebra whose K-theory we take. My final goal is a model for materials with disorder based on the Roe C^*-algebra, following [3].

Further reading

[2] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, *Quantum spin Hall effect and topological phase transition in* HgTe *quantum wells*, Science **314** (2006), 1757–1761.

[3] E. Ewert and R. Meyer, *Coarse geometry and topological phases*, Comm. Math. Phys. (2018), accepted. arXiv:1802.05579.

Constanza Rojas-Molina (University of Düsseldorf): Random Schrödinger operators.

Abstract We will give an introduction to the theory of random Schrödinger operators, by studying the Anderson model on the d-dimensional lattice. This model was first proposed by P. W. Anderson in the late 50s to explain the absence of wave propagation in materials with impurities. The Anderson model is a self-adjoint Schrödinger operator with a random potential that represents the medium. As a consequence of the presence of disorder, waves associated with low energies remain localized in space as time evolves, and the operator exhibits pure point spectrum in that region of its spectrum. This implies that electronic transport is suppressed and therefore, in that energy range, the material behaves as an insulator. This phenomenon is known as Anderson localization. The aim of this course is to give a panoramic view on the field of random Schrödinger operators, and give a proof of Anderson localization in a simple case, using the Fractional Moment Method, also known as Aizenman-Molchanov method. We will follow closely the lecture notes of G. Stolz [7], and refer to [4–6] for further reading.

Further reading

[4] M. Aizenman and S. Warzel, *Random operators: Disorder effects on quantum spectra and dynamics*, Grad. Stud. Math., vol. 168, Amer. Math. Soc., 2016.

[5] W. Kirsch, *An invitation to random Schrödinger operators*, with an appendix by F. Klopp, Random Schrödinger operators (M. Disertori, W. Kirsch, A. Klein, F. Klopp, and V. Rivasseau, eds.), Panor. Synthèses, vol. 25, Soc. Math. France, Paris, 2008, pp. 1–119. arXiv:0709.3707.

[6] C. Rojas-Molina, *Random Schrödinger operators on discrete structures,* lecture notes, Panor. Synthèses, Soc. Math. France, Paris, to appear. arXiv:1710.02293.

[7] G. Stolz, *An introduction to the mathematics of Anderson localization*, Arizona School of Analysis and Applications, Entropy and the quantum II, Contemp. Math., vol. 551, 2011. arXiv:1104.2317.

Hermann Schulz-Baldes (University of Erlangen-Nuremberg): K-theory and topological insulators.

Abstract Topological insulators are solid state systems of independent electrons for which the Fermi level lies in a mobility gap, but the Fermi projection is nevertheless topologically non-trivial, namely it cannot be deformed into that of a normal insulator. This non-trivial topology is encoded in adequately defined invariants and implies the existence of surface states that are not susceptible to Anderson localization. The talks report on recent progress in the understanding of the underlying mathematical structures, with a particular focus on index theory.

Further reading

[8] E. Prodan and H. Schulz-Baldes, *Bulk and boundary invariants for complex topological insulators. From K-theory to physics*, Math. Phys. Stud., Springer, Cham, 2016. arXiv:1510.08744.

Ingo Witt (University of Göttingen): Spectral theory.

Abstract We will provide a brief introduction to the spectral theory of bounded self-adjoint operators on Hilbert space. Spectral theory constitutes a far-reaching generalization to infinite dimensions of the fact that a Hermitian matrix has real eigenvalues and that its eigenvectors can be chosen to form an orthonormal basis. The spectrum of a bounded self-adjoint operator can be more complicated than just consisting of isolated eigenvalues of finite multiplicity (that is, discrete spectrum), a circumstance which constitutes the main challenge. Eventually, spectral theory culminates in the spectral theorem which we will state in its different guises.

We will carefully introduce basic concepts, learn how to deal with the difficulties connected with the nature of spectrum, and also discuss some instructive examples.

Further reading

[9] M. Reed and B. Simon, *Methods of modern mathematical physics. I: Functional analysis*, Academic Press, New York, 1980.

Short Presentations

Wednesday

16:10–16:30	**Lasha Baramidze**: Uniform convergence of double Vilenkin-Fourier series.
16:35–16:55	**Simon Becker**: Graphene in magnetic fields.

Abstracts

Lasha Baramidze (Tbilisi State University): Uniform convergence of double Vilenkin-Fourier series.

Abstract: In this talk, we will discuss the uniform convergence problem for rectangular partial sums of double Fourier series on bounded Vilenkin groups for functions of partially bounded oscillation.

Simon Becker (University of Cambridge): Graphene in magnetic fields.

Abstract: Using semiclassical (with the strength of the magnetic field as the small parameter) and spectral methods we study the properties of graphene in magnetic fields with a simple quantum graph model. From the semiclassical analysis, we obtain a geometric description of the density of states which can be used to study magnetic oscillations such as the de Haas-van Alphen effect. We finally provide an outlook on 2D models for graphene in a magnetic field and related configurations such as bilayer graphene.

This talk is based on joint work with M. Zworski, as well as R. Han and S. Jitomirskaya.

Schedule

	Mon	**Tue**	**Wed**	**Thu**	**Fri**
8:30	Registration				
9:15–10:00	Bakuradze	Rojas-Molina	Schulz-Baldes	Graf	Meyer
10:10–10:55	Witt	Bakuradze	Opening VIAM conference	Rojas-Molina	Schulz-Baldes
	Coffee	**Coffee**	**Coffee**	**Coffee**	**Coffee**
11:25–12:10	Meyer	Witt	Rojas-Molina	Schulz-Baldes	Graf
	Lunch	**Lunch**	**Lunch**	**Lunch**	**Lunch**
14:00–14:45	Bakuradze	Graf	Meyer	**Excursion**	Rojas-Molina
14:55–15:40	Rojas-Molina	Schulz-Baldes	Q&A		Q&A
	Coffee	**Coffee**	**Coffee**		**Coffee**
16:10–16:55	Graf	Bakuradze	Short presentations		Meyer
17:05–17:50	Schulz-Baldes	Witt	Graf		
18:00	Dinner	Dinner	Dinner	(19:00) Conference dinner	Dinner